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ON THE EXISTENCE OF HERMITE-BIRKHOFF QUADRATURE FORMULAS OF GAU--ETC(U)
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ON THE EXISTENCE OF HERMITE-BIRKHOFF QUADRATURE FORMULAS OF GAUSSIAN TYPE

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ON THE EXISTENCE OF HERMITE-BIRKHOFF QUADRATURE FORMULAS OF GAUSSIAN TYPE

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ABSTRACT

Existence of quadrature formulas ~~of~~ Gaussian type related to Hermite-Birkhoff interpolation is proved, for a class of incidence matrices satisfying the conditions of the Atkinson-Sharma Theorem. For the subclass of Hermite matrices this analysis furnishes yet another proof of the existence of Gaussian quadrature formulas with multiple nodes.

AMS (MOS) Subject Classifications: 41A05, 41A55.

Key Words: Gaussian quadrature formulas, Hermite-Birkhoff interpolation, Tchebychev systems.

Work Unit Number 6 - Spline Functions and Approximation Theory.

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SIGNIFICANCE AND EXPLANATION

Methods for approximating the integral of a function, given the values of the function and/or some of its derivatives at several points in the interval of integration, are investigated. The integral is approximated by a weighted sum of the given data--a quadrature formula.

It is shown that for a wide class of different data configurations, there exist appropriate points of evaluation and weights such that the resulting quadrature formula is exact for all polynomials of the maximal possible degree. The well-known Gaussian quadrature formulas represent the particular case in which only function values (and no derivatives) are employed.

ON THE EXISTENCE OF HERMITE-BIRKHOFF QUADRATURE FORMULAS OF GAUSSIAN TYPE

Nira Dyn*

1. Introduction and Preliminaries

This paper studies the existence of quadrature formulas of Gaussian type related to Hermite-Birkhoff interpolation problems. Given the $m \times n$ incidence matrix $E = (e_{ij})_{i=1,}^{m} {}_{j=0}^{n-1}$ with entries consisting of zeros and ones, having precisely N ones, $N < n$, we are interested in the existence of a quadrature formula of the form:

$$\int_a^b f\,d\sigma = \sum_{e_{ik}=1} a_{ik} f^{(k)}(x_i) \quad , \quad a \le x_1 < x_2 < \cdots < x_m < b \quad , \tag{1.1}$$

which is exact for Π_{n-1} -the space of all polynomials of degree $\le n-1$. Incidence matrices with $N < n$ ones which admit such quadrature formulas are termed in [3] "matrices of Gaussian type." It is proved in [3] that the quadrature formula (1.1) can be exact for Π_{n-1} only if $n \le N - k$, where k is the minimal number of ones which must be added to E to obtain a matrix without odd sequences in rows corresponding to interior points of $[a,b]$. Two classes of matrices of Gaussian type, admitting quadrature formulas (1.1) exact for Π_{n-1}, with $n = N - k$, are known:

(a) The class of Hermite matrices (matrices consisting of sequences of ones starting at column 0 -Hermite sequences) with all the sequences corresponding to points in (a,b) of odd order [2], [6].

(b) The class of incidence matrices derived from quasi-Hermite matrices with Hermite sequences of length 2 in rows $2,\cdots,m-1$, by changing the last one in each of these sequences into zero [11]. (The definition of quasi-Hermite matrices is given at the end of this section).

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In the following we characterize a wide class of incidence matrices of Gaussian type for which $n = N - k$, where k equals the number of Hermite sequences corresponding to points in (a,b). This class contains the classes (a), (b). Our analysis is based on the Atkinson-Sharma Theorem [1] and on the existence of the classical Gaussian quadrature formulas (termed also principal representations) for Chebychev systems [8]. The idea of proof is in some respect an extension of the approach of Markov [10] to the construction of the Gaussian quadrature formulas (GQF) in the classical sense. In this approach the GQF is derived by integration of the Hermite interpolation polynomial:

$$P_{2k-1}(t;X^*) = \sum_{i=1}^{k} [f(x_i^*)p_{i0}(t;X^*) + f'(x_i^*)p_{i1}(t;X^*)] \quad ,$$

$$p_{ij} \in \Pi_{2k-1} \ , \quad p_{ij}^{(s)}(x_r) = \delta_{ir}\delta_{js} \quad s = 0,1, \ r = 1,\cdots,k, \ j = 0,1, \ i = 1,\cdots,k,$$

with $X^* = \{a \leq x_1^* < \cdots < x_k^* \leq b\}$ chosen so that

$$\int_a^b p_{i1}(t;X^*)dt = 0 \quad , \quad i = 1,\cdots,k \quad .$$

Although our method of proof does not yield uniqueness of the quadrature (1.1) even for matrices of class (a), for which uniqueness is known [2], [6], yet the proof of existence is somewhat simpler than the proofs in [2], [6]. Moreover by the same method it is possible to extend the uniqueness result to the case of quadrature formulas related to quasi-Hermite matrices.

The results and proofs are stated for polynomials, but the extension to extended Chebychev systems is straightforward. (See §3).

In Section 2 we prove the existence of quadrature formulas of Gaussian type related to a certain class of incidence matrices. Section 3 consists of remarks on some extensions and on certain interesting specific cases.

We conclude this section by introducing notations and citing some results from the theory of Hermite-Birkhoff interpolation. Let $AS_{m \times n}$ denote the class of incidence matrices $E = (e_{ij})_{i=1,\ j=0}^{m,\ n-1}$ satisfying the following conditions:

$$(1.2)\qquad \sum_{j=0}^{n-1}\sum_{i=1}^{m} e_{ij} = n\ ,\qquad \sum_{j=0}^{s-1}\sum_{i=1}^{m} e_{ij} \geq s\ ,\quad s = 1,\cdots,n-1 \quad \text{(Pólya conditions)}$$

(1.3) All the non-Hermite sequences in rows $2,\cdots,m-1$ are even.

(A sequence is a maximal string of ones in a row).

The well-known theorem of Atkinson-Sharma [1] states that the interpolation problem at (E,X):

$$(1.4)\qquad p^{(j)}(x_i) = \alpha_{ij}\ ,\quad e_{ij} = 1\ ,\quad p \in \Pi_{n-1}$$

has a unique solution for any $X = \{a \leq x_1 < x_2 < \cdots < x_m \leq b\}$ and any data $\{\alpha_{ij},\ e_{ij} = 1\}$ (E is order-poised) if E satisfies (1.2) and if E contains no sequence $e_{i,k+1} = e_{i,k+2} = \cdots = e_{i,k+r} = 1$ with r odd, such that $e_{\nu\mu} = 1$ for some $\nu < i$, $\mu \leq k$ and for some $\nu > i$, $\mu \leq k$.

The interpolation problem (1.4) for $X = \{a \leq x_1 < x_2 < \cdots < x_m \leq b\}$ can be extended continuously to any $X = \{a \leq x_1 \leq x_2 \leq \cdots \leq x_m \leq b\}$, by considering the problem (1.4) for $(\hat{E},\hat{X})$, where $\hat{X} = \{a \leq \hat{x}_1 < \hat{x}_2 < \cdots < \hat{x}_\ell \leq b\}$ consists of all the distinct nodes of X, and $\hat{E}$ is obtained from E by coalescing rows of E corresponding to equal nodes of X, according to the rule [5], [4]:

$$(1.5)\qquad \text{For } \hat{x}_s = x_i = \cdots = x_{i+k},\ \hat{e}_{sj} = 1 \text{ if and only if } \sum_{\mu=r}^{j}\sum_{\nu=i}^{i+k} e_{\nu\mu} \geq j - r + 1 \text{ for some } 0 \leq r \leq j$$

The Atkinson-Sharma theorem implies that all AS matrices are order-poised. A subclass of particular interest of the AS matrices are the quasi-Hermite matrices - matrices satisfying (1.2) and containing only Hermite sequences in rows $2,\cdots,m-1$.

A quadrature formula of the form (1.1), exact for Π_{n-1}, is termed hereafter "Hermite-Birkhoff Gaussian quadrature formula" (HB-GQF) if

$$(1.6)\qquad N = \sum_{j=0}^{n-1}\sum_{i=1}^{m} e_{ij} = n - \sum_{\{i \mid x_i \in (a,b)\}} e_{i0}$$

2. Existence of Hermite-Birkhoff Gaussian Quadrature Formulas

A key observation in the forthcoming analysis is the following direct result of the Atkinson-Sharma Theorem (see also [9] p. 89):

<u>Lemma 1</u>: Let E be an $m \times n$ incidence matrix satisfying:

$$e_{i0} = 0 \qquad 1 \le i \le m \tag{2.1}$$

$$\sum_{j=0}^{s-1} \sum_{i=1}^{m} e_{ij} \ge s - r \ , \quad s = 1,2,\cdots,n-2 \ , \quad \sum_{j=0}^{n-1} \sum_{i=1}^{m} e_{ij} = n - r \tag{2.2}$$

for some positive integer r. If all sequences of E are even then for any $X = \{a \le x_1 < x_2 < \cdots < x_m \le b\}$ the subspace

$$P_0(E,X) = \{p \mid p \in \Pi_{n-1} \ , \ p^{(j)}(x_i) = 0 \ , \ e_{ij} = 1\} \tag{2.3}$$

is a Chebychev space of dimension r on $[a,b]$.

<u>Remark 1</u>: For $X = \{a \le x_1 \le x_2 \le \cdots \le x_m \le b\}$ the space $P_0(E,X)$ is defined by $P_0(\hat{E},\hat{X})$ where $\hat{X} = \{a \le \hat{x}_1 < \cdots < \hat{x}_\ell \le b\}$ consists of the distinct components of X, and where $\hat{E}$ is obtained from E by coelescing rows of E corresponding to equal components of X as in (1.5).

If E satisfies the condition of Lemma 1 so does $\hat{E}$, and therefore $P_0(E,X)$ is a Chebychev space of dimension r on $[a,b]$, for all X in the m dimensional symplex

$$S^m = \{X \mid X = \{a \le x_1 \le \cdots \le x_m \le b\}\} \ . \tag{2.4}$$

Moreover $P_0(E,X)$ depends continuously on X.

Similarly we have

<u>Lemma 2</u>: Let E be an $m \times n$ incidence matrix satisfying (2.1) and (2.2). If all sequences of E in rows $2,\cdots,m-1$ are even, then for any $X = \{a = x_1 \le x_2 \le \cdots \le x_{m-1} \le x_m = b\}$ the subspace $P_0(E,X)$ is a Chebychev space of dimension r, which depends continuously on X.

Analogous results hold if all rows of E except the first (the last) consist of even sequences only. In this case $x_1 = a$ ($x_m = b$) and the rest of the nodes vary in $[a,b]$ appropriately.

The main result of this work is the content of the next theorem.

Theorem 1: Let $E \in AS_{mxn}$ contain only even sequences, k of which are Hermite sequences, $k > 0$. Let the matrix E^* be derived from E by replacing the last one in each Hermite sequence of E by zero. Then E^* admits a HB-GQF with nodes $X^* = \{a < x_1^* < x_2^* < \cdots < x_m^* < b\}$, for any positive measure which is supported on more than k points in (a,b).

Proof: Let

$$I_E = \{i \mid 1 \le i \le m \ , \ e_{i0} = 1\} \equiv \{i_1 < i_2 < \cdots < i_k\} \ ,$$

(2.5)

$$I_E^c = \{i \mid 1 \le i \le m \ , \ e_{i0} = 0\} \ ,$$

$$\mu_i = \max\{j \mid e_{is} = 1 \ \text{ for all } \ 0 \le s \le j\} \ \text{ if } \ i \in I_E \ ,$$

(2.6)

$$\mu_i = -1 \ \text{ if } \ i \in I_E^c \ ,$$

and let $\tilde{E} = (\tilde{e}_{ij})_{i=1,}^{m} \,_{j=0}^{n-1}$ be obtained from E by replacing the first and last ones in each Hermite sequence of E by zero: $\tilde{e}_{i0} = \tilde{e}_{i\mu_i} = 0$, $i \in I_E$, otherwise $\tilde{e}_{ij} = e_{ij}$. Since $E \in AS_{mxn}$ and has only even sequences, the matrix $\tilde{E}$ satisfies the conditions of Lemma 1, and $\sum_{i=1}^{m} \sum_{j=0}^{n-1} \tilde{e}_{ij} = n - 2k$. Hence by Lemma 1 and Remark 1, $P_0(\tilde{E},X)$ is a Chebychev subspace of dimension $2k$ for any $X \in S^m$.

Using the result about the existence of a Gaussian quadrature formula (lower principal representation) for a Chebychev space [8], we conclude the existence of unique $Z = \{a < z_1 < \cdots < z_k < b\}$ and $w_i > 0$, $i = 1,\cdots,k$, such that

$$\int_a^b p \, d\sigma = \sum_{i=1}^{k} w_i \, p(z_i) \quad , \quad p \in P_0(\tilde{E},X) \tag{2.7}$$

for any positive measure $d\sigma$ supported on more than k points of (a,b).

In the following we construct a continuous mapping of the symplex S^k into itself. Let T_k^m be any continuous mapping from S^k into S^m such that for $Y = \{a < y_1 < \cdots < y_k < b\}$, $X = T_k^m Y$ is of the form:

$$x_{i_j} = y_j \ , \ j = 1,\cdots,k \ , \ a < x_1 < x_2 < \cdots < x_m < b \ . \tag{2.8}$$

In case $k = m$ (all rows of E contain Hermite sequences) T_k^m is the identity mapping. For $X \in S^m$ let $G_m^k X = Z = \{a < z_1 < \cdots < z_k < b\} \in S^k$ with $z_1,\cdots,z_k$ the points of the GQF (2.7). The mapping $T \equiv G_m^k T_k^m$ is a continuous mapping of S^k into itself, and therefore by Brouwer fixed-point theorem [12] there exists $Y^* \in S^k$ such that

$$T\,Y^* = Y^*$$

Moreover, since for all Y, $T\,Y$ is an interior point of S^k, $Y^* = \{a < y_1^* < \cdots < y^*_k < b\}$, and therefore $X^* = T_k^m Y^*$ is of the form $X^* = \{a < x_1^* < \cdots < x_m^* < b\}$. We conclude the proof of the theorem by showing that E^* admits an HB-GQF at the nodes X^*.

Since E is order poised, for any $e_{ij} = 1$ there exists a unique polynomial p_{ij} satisfying

$$\frac{d^r}{dx^r} p_{ij}(x_s^*) = \delta_{is}\delta_{jr} \ , \ e_{rs} = 1 \ , \ p_{ij} \in \Pi_{n-1} \ , \tag{2.9}$$

and any $p \in \Pi_{n-1}$ can be written as

$$p(t) = \sum_{e_{ij}=1} p^{(j)}(x_i^*)\, p_{ij}(t) \tag{2.10}$$

Moreover since $T\,Y^* = Y^*$, the points $x_{i_1}^*,\cdots,x_{i_k}^*$ are the nodes of a GQF of the form (2.7) for $P_0(\hat{E},X^*)$. Now $p_{\ell\mu_\ell} \in P_0(\hat{E},X)$ for $\ell \in I_E$, and therefore

$$\int_a^b p_{\ell\mu_\ell}\, d\sigma = \sum_{s=1}^k w_s^* p_{\ell\mu_\ell}(x_{i_s}^*) = 0 \ , \ \ell \in I_E \ . \tag{2.11}$$

Integrating (2.10) and using (2.11), we obtain

$$(2.12)\qquad \int_a^b p\,d\sigma = \sum_{\substack{e_{ij}=1\\ j\neq\mu_i}} a_{ij}\,p^{(j)}(x_i^*) = \sum_{e_{ij}^*=1} a_{ij}\,p^{(j)}(x_i^*)\quad,\quad p\in\Pi_{n-1}$$

with

$$(2.13)\qquad a_{ij} = \int_a^b p_{ij}\,d\sigma\quad,\quad e_{ij}^* = 1\ .$$

<u>Corollary 1</u>: In the HB-GQF (2.12), $a_{ij} > 0$ for $i \in I_E$, j even $j < \mu_i$.

<u>*Proof*</u>: For $i \in I_E$, j even $j < \mu_i$, let q_{ij} be the polynomial solving the following interpolation problem:

$$(2.14)\qquad q_{ij}^{(s)}(x_\nu^*) = 0\quad,\quad e_{\nu s} = 1\quad,\quad \nu \neq i$$

$$(2.15)\qquad q_{ij}^{(s)}(x_i^*) = 0\quad,\quad e_{is} = 1\quad,\quad s \neq j,\ \mu_i$$

$$(2.16)\qquad q_{ij}^{(j)}(x_i^*) = 1\quad,$$

$$(2.17)\qquad q_{ij}(a) = 0$$

Then by (2.12) $\int_a^b q_{ij}\,d\sigma = \sum_{e_{\nu s}^*=1} a_{\nu s}\,q_{ij}^{(s)}(x_\nu^*) = a_{ij}$

To prove that $a_{ij} > 0$ it is sufficient to show that $q_{ij} \geq 0$, $q_{ij} \not\equiv 0$ on the support of $d\sigma$. We first show that $q_{ij}(x) \neq 0$, $x \in (a,b) - \{x_\nu^* \mid \nu \in I_E\}$, and that the zeros of q_{ij} in $\{x_\nu^* \mid \nu \in I_E\}$ are even. Suppose to the contrary that either $q_{ij}(\xi) = 0$ for $\xi \in (a,b) - \{x_\nu^* \mid \nu \in I_E\}$ or that $q_{ij}^{(\mu_\nu+1)}(x_\nu^*) = 0$ for some $\nu \in I_E$, $\nu \neq i$. Then q_{ij} is a nontrivial solution of a homogeneous interpolation problem, corresponding to a matrix $\bar{E}$ obtain from E by substituting $\bar{e}_{ij} = 0$, $\bar{e}_{i\mu_i} = 0$, adding a row with 1 in the first column corresponding to the point a, and either adding a row with 1 in its first column corresponding to the point ξ, or substituting $\bar{e}_{\nu,\mu_{\nu+1}} = 1$. In either cases $\bar{E} \in AS_{mxn}$ - a contradiction. Thus q_{ij} vanishes in (a,b), only at the points $\{x_\nu^* \mid \nu \in I_E\}$, and all these zeros are even by (2.14) - (2.16) and the structure of E. In view of (2.16) $q_{ij} \geq 0$, while $q_{ij} \not\equiv 0$ on the support of $d\sigma$ since $\{x_\nu^* \mid \nu \in I_E\}$ contains k points.

The same idea of proof can be extended to all the matrices in $AS_{m\times n}$ having even Hermite sequences in rows corresponding to interior points of (a,b). We state the result and sketch the proof for one such case:

<u>Theorem 2</u>: Let $E \in AS_{m\times n}$ contain $k > 0$ even Hermite sequences in rows $2,\cdots,m-1$ and no odd Hermite sequence in these rows. Then E^*, derived from E by replacing the last one in every Hermite sequence in rows $2,\cdots,m-1$ by zero, admits an HB-GQF with nodes $X^* = \{a = x_1^* < x_2^* < \cdots < x_m^* = b\}$, for any positive measure which is supported on more than k points of (a,b).

<u>Proof</u>: Define I_E, I_E^C, μ_i as in (2.5), (2.6), $\bar{I}_E = \{i \mid i \in I_E, i \neq 1, m\} \equiv \{i_1 < \cdots < i_k\}$, and denote by $\hat{E}$ the matrix obtained from E by substituting

$$\hat{e}_{i0} = 0 \quad , \quad i \in I_E \quad , \quad \hat{e}_{i\mu_i} = 0 \quad , \quad i \in \bar{I}_E \ .$$

For $X = \{a = x_1 \leq x_2 \leq \cdots \leq x_{m-1} \leq x_m = b\}$ let $P_0(\hat{E},X)$ be as in Lemma 2. By construction of $\hat{E}$, for all $(x_2,\cdots,x_{m-1}) \in S^{m-2}$, $P_0(\hat{E},X)$ is a Chebychev space of dimension $2k + e_{10} + e_{m0}$. We define a mapping T from S^k into S^k as follows: For $Y \in S^k$ let $X = \{a = x_1 \leq \cdots \leq x_m = b\}$, with $x_2,\cdots,x_{m-1}$ defined as in (2.8). Then $T\,Y = Z \in S^k$ where $Z = \{a < z_1 < z_2 < \cdots < z_k < b\}$ are the interior points of the GQF (principal representation) for $P_0(\hat{E},X)$, corresponding to the measure $d\sigma$, which involves a if $e_{10} = 1$ and b if $e_{m0} = 1$. Using the fixed point of T, Y^*, we construct $x_2^* < \cdots < x_{m-1}^*$ by (2.8), and proceed as in the proof of Theorem 1 to construct the HB-GQF corresponding to E^* at the nodes $X^* = \{a = x_1^* < x_2^* < \cdots < x_m^* = b\}$.

As in the case of Theorem 1 we have:

<u>Corollary 2</u>: Let E^* be defined as in Theorem 2, and let

$$\int_a^b p\, d\sigma = \sum_{e_{ij}^*=1} a_{ij}\, p^{(j)}(x_i^*) \quad , \quad p \in \pi_{n-1} \tag{2.18}$$

be the corresponding HB-GQF. Then $a_{ij} > 0$ for $i \in \bar{I}_E$, j even, $j < \mu_i$. Moreover $a_{1j} > 0$ for $j \leq \mu_1$, $(-1)^j a_{mj} > 0$ for $j \leq \mu_m$.

3. Extensions and Remarks

3.1 Let E in Theorem 2 be a quasi-Hermite matrix with $\mu_i = 1$, $i = 2,\cdots,m-1$. Then $\hat{E}$ has non-zero elements only in the first and last rows, $\sum_{j=0}^{n-1} (e_{1j} + e_{mj}) = n - 2(m-2)$, and for all $X = \{a = x_1 \le \cdots \le x_m = b\}$

$$(3.1)\quad \begin{aligned} P_0(\hat{E},X) = P_0(\hat{E}) = \{p \mid p \in \Pi_{n-1} \ , \ p^{(j)}(a) = 0 \ , \ e_{1j} = 1 \ , \ j > 0 \ , \\ p^{(j)}(b) = 0 \ , \ e_{mj} = 1 \ , \ j > 0\} \end{aligned}$$

is a Chebychev space of dimension $d = 2(m-2) + e_{10} + e_{m0}$. Since the space $P_0(\hat{E},X)$ is independent of $x_2,\cdots,x_{m-1}$, the points $x_2^*,\cdots,x_{m-1}^*$ of Theorem 2 are the interior points of the GQF for $P_0(\hat{E})$ involving a if $e_{10} = 1$ and involving b if $e_{m0} = 1$. Thus in this special case the derivation of the HB-GQF for E^* does not involve the construction of a fixed point of a mapping. In addition the same arguments yield the uniqueness of the HB-GQF corresponding to E^*.

3.2 A direct consequence of the simple relation between the HB-GQF constructed in 3.1 for the matrix E^*, and a certain GQF corresponding to $P_0(\hat{E})$, is the following extremal property of this HB-GQF:

Let E be as in 3.1. Among all polynomials from

$$(3.2)\quad Q_n = \{q \mid q \in \Pi_n \ , q \ge 0 \ , q^{(j)}(a) = 0 \ , e_{1j} = 1 \ , q^{(j)}(b) = 0 \ , e_{mj} = 1\}$$

with leading coefficient $(-1)^s$, $s = \sum_{j=0}^{n-1} e_{mj}$, the one which has double roots at $x_2^*,\cdots,x_{m-1}^*$ of 3.1 minimizes $\int_a^b q(x)dx$.

To see this observe that any $q \in Q_n$ with leading coefficient $(-1)^s$, can be written as $q = p_n - p$, where $p_n = (-1)^s x^n + \cdots$ satisfies

$$(3.3)\quad p_n^{(j)}(a) = 0 \ , \ e_{1j} = 1 \ , \ j > 0 \ , \ p_n^{(j)}(b) = 0 \ , \ e_{mj} = 1 \ , \ j > 0 \ ,$$

and where $p \in P_0(\hat{E})$ satisfies $p \le p_n$ on $[a,b]$, $p(a) = p_n(a)$ if $e_{10} = 1$,

$p(b) = p_n(b)$ if $e_{m0} = 1$. Since $\{p_n\} \cup P_0(\hat{E})$ is a Chebychev space of dimension $d+1$, the extremal property of the GQF used in 3.1 yields the required result. (For extremal properties of GQF (principal representations) consult [8]). The results of 3.1 and 3.2 are the content of Theorem 4 in [11]. The proof of these results as sketched here, seems to be simpler.

It should be noted that GQF with multiple nodes (HB-GQF admitted by Hermite matrices) have a similar extremal property [7].

3.3 Theorems 1, 2 can be extended by the same method of analysis to the case of extended Chebychev systems $(u_1,\cdots,u_n \in C^{n-1}[a,b]$ is an extended Chebychev system if any nontrivial "polynomial" $\sum_{i=1}^{n} a_i u_i$ has at most $n-1$ zeros counting multiplicities). The Atkinson-Sharma Theorem is valid also for extended Chebychev systems [5], but with the operators $\frac{d^k}{dx^k}$ $k = 1,\cdots,n-1$, replaced by certain differential operators $D_1,\cdots,D_{n-1}$ related to the extended Chebychev system. In case of Hermite matrix E^*, the resulting HB-GQF involves only evaluations of the function and its derivatives of order at most $n-1$. Otherwise the HB-GQF is of the form:

(3.4)
$$\sum_{\substack{e^*_{ij}=1 \\ j\le\mu^*_i}} a_{ij} f^{(j)}(x^*_i) + \sum_{\substack{e^*_{ij}=1 \\ j>\mu^*_i}} a_{ij} (D_j f)(x^*_i)$$

where $\mu^*_i + 1$ is the number of ones in the Hermite sequence in row i of E^*. $(\mu^*_i = -1$ if $e^*_{i0} = 0)$.

3.4 The existence and uniqueness of the HB-GQF admitted by an Hermite matrix E^*, in case of extended Chebychev systems, is proved in [2], [6]. Using the uniqueness of GQF with multiple nodes, we can prove the uniqueness of the HB-GQF related to a quasi-Hermite matrix by a construction similar to 3.1.

Indeed for E quasi-Hermite let ν_i be defined as in Theorem 2, and let

(3.5) $\hat{P}_0(E) = \{p \mid p \in \Pi_{n-1}, p^{(j)}(a) = 0, e_{1j} = 1, j > 0, p^{(j)}(b) = 0, e_{mj} = 1, j > 0\}$.

By Lemma 2 $\hat{P}_0(E)$ is a Chebychev space of dimension $n - \sum_{j=1}^{n-1} (e_{0j} + e_{mj})$. Now

$$\int_a^b p \, d\sigma = \sum_{i=2}^{m-1} \sum_{j=0}^{\nu_i - 1} a_{ij} p^{(j)}(x_i^*) + \sum_{e_{1j}=1} a_{1j} p^{(j)}(a) + \sum_{e_{mj}=1} a_{mj} p^{(j)}(b) ,$$

for all $p \in \Pi_{n-1}$ if and only if $x_2^*, \cdots, x_{m-1}^*$ are the interior nodes of the unique GQF with multiple nodes for $\hat{P}_0(E)$, which involves a if $e_{10} = 1$ and involves b if $e_{m0} = 1$.

3.5 It is conjectured that for E^* in Theorem 1 with $k = m$ (all rows contain Hermite sequences) the corresponding HB-GQF is unique, as in the case of Hermite matrices [2], [6]. Uniqueness cannot be expected in the more general case due to the arbitrariness in the construction of the mapping T_k^m of (2.8).

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Existence of quadrature formulas of Gaussian type related to Hermite-Birkhoff interpolation is proved, for a class of incidence matrices satisfying the conditions of the Atkinson-Sharma Theorem. For the subclass of Hermite matrices this analysis furnishes yet another proof of the existence of Gaussian quadrature formulas with multiple nodes.

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